# Unorthodox Thoughts about Deformation, Elasticity, and Stress

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The nature of elastic deformation is examined in the light of the potential theory. The concepts and mathematical treatment of elasticity and the choice of equilibrium conditions are adopted from the mechanics of discrete bodies, e. g., celestial mechanics; they are not applicable to a change of state. By nature, elastic deformation is energetically a Poisson problem since the buildup of an elastic potential implies a change of the energetic state in the sense of thermodynamics. In the Euler-Cauchy theory, elasticity is treated as a Laplace problem, implying that no change of state occurs, and there is no clue in the Euler-Cauchy approach that it was ever considered as one. The Euler-Cauchy theory of stress is incompatible with the potential theory and with the nature of the problem; it is therefore wrong. The key point in the understanding of elasticity is the elastic potential.

*Key words:* Potential Theory; Elasticity; Stress; Poisson Equation; Cauchy Theory.

## Introduction

Despite much effort to carefully outline the principles of a physical theory, one cannot always avoid to get into circular reasoning. For example, in mechanics the Newtonian definition of a force, Newtonian work, the Hamiltonian, the use of an equation of motion and the continuity equation are interdependent concepts; if one accepts one of them as relevant to the problem unter discussion, the others must necessarily follow. It is not really possible to consider other concepts then. This can only be avoided if the nature of that physical process is carefully examined before the above concepts are even considered; the theory by which this is done is the theory of potentials which occupies the highest level of generality in the physical sciences. Any physical process must fit into one of the categories offered by that theory.

I have never seen an attempt to derive the theory of elasticity from the theory of potentials. Historically, the stress theory is older (Euler published the cut model in 1776, Cauchy proposed the stress tensor in the first decade of the 19th century; the theory of potentials was developed between 1830 - 1840 by Gauss and contemporaries), and – insight in hindsight – was heavily influenced by the availability of Hooke's law, which is phenomenological. I believe that it was not possible to ask the right questions regarding the nature of elasticity and deformation before 1842, the year in which both the first law of thermodynamics and the Hamiltonian were published, from whence it became possible to make a clear distinction between the conservative physics of discrete bodies in free space on the one hand, and non-conservative physics of changes of state on the other. Fundamental to the understanding of the potential theory is the concept of vector fields, first proposed by Lagrange in 1784, one year after Euler's death. The fact that Euler did not know it yet explains many idiosyncrasies of continuum mechanics.

Over the years I have had the opportunity to realize that the concepts, geometric and mathematical properties of the potential theory are so completely unknown in continuum mechanics that a question regarding the nature of the elastic potential was usually met with silence; the reason is that the conclusions to be drawn from potential theory are so obviously at variance with continuum mechanics that only one of them can be correct. Thus the first step in this paper is a discussion of the principles of continuum mechanics and their hidden consequences, to show which questions are implicitly asked and answered by choosing a particular concept. This discussion cannot avoid being repetitive, due to the aforementioned interdependence of thoughts. It follows a short outline of the theory of potentials. Finally, the simplest of all elastic defor-

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mations, isotropic loading, is presented in vector field form. Notation: $\boldsymbol{f}$ = vectors, $f$ = vector magnitudes, $\mathbf{T}$ = tensors, $S$ = scalars and points. References are often given with page numbers, e. g., [1:147], in order to facilitate comparison.

## The Tools of the Euler-Cauchy Theory

### *1. What is Deformation?*

The answer provided by the textbooks is: "Deformation is a transformation of points" [2]. The phenomenological character of present continuum mechanics cannot possibly be expressed in more succinct form. – Deformation is a physical process. As such it may be a conservative or a non-conservative process; if it is the latter, it is still to be determined whether it is a reversible or an irreversible process. (A conservative process is reversible by definition. Henceforth the term "reversible" will be used only in contrast to "irreversible", i. e. it is implied that the process is non-conservative.)

A conservative system is a region in space the interior of which does not interact with its surrounding, especially not in form of exchange of heat or work; it is an isolated system. Strictly speaking, a conservative process will take place only inside such a system, and ideally it cannot be measured because observations require interaction with the exterior. The characteristic property of such a process is that it will not change the energy $U$ of the system. The mathematical tool for a proper approach is the Laplace equation, $\nabla^2 U = 0$ [1:121; 15].

If interaction takes place between the interior and the exterior, the process is strictly no longer conservative even if the energetic state of the system is unchanged; the system acts as a source and as a sink of fluxes simultaneously. The mathematics of conservative processes may still be applied, provided they are found to be applicable; this point must be checked against the nature of the process under investigation. For example, a steady-state heat flow through a system which remains itself in a constant thermal state, can be approached through the Laplace equation because heat can be transmitted only radially, and the exclusion of non-radial components from consideration which is implied by the Laplace equation is justified.

If the energetic state of the system is changed due to interaction of interior and exterior, the process is non-conservative (its energetic state is not conserved); the equation to be used is the Poisson equation $\nabla^2 U = \varphi$, where $\varphi$ is the charge [1:156; 15]. The process is reversible if the entropy flux is zero; otherwise it is irreversible.

A consideration of deformation without consideration of its physical characteristics according to the above categories must be phenomenological because it may be a conservative, a reversible, or an irreversible process. 1. Volume-constant, infinitely slow equilibrium flow of a real gas is a conservative process because no net work is done on the system, hence no elastic potential builds up. 2. Elastic deformation is reversible. 3. Plastic flow and flow involving diffusion, i. e. viscous flow, are irreversible. In cases 2. and 3. the Laplace equation is not applicable. The key to the understanding of stress and deformation is the elastic state because *by its nature, elastic deformation represents a change of state.*

### *2. Definition of Force and Density*

Euler gave the equations of motion as [3:151, Eqn. III.1-1]

$$\begin{aligned} &\int \rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \mathrm{d}V = \boldsymbol{f}, \\ &\int (\boldsymbol{x} - \boldsymbol{x}_0) \times \rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \mathrm{d}V = \boldsymbol{F}. \end{aligned} \tag{1}$$

Newton defined a force as $\boldsymbol{f} = m\,\boldsymbol{a}$, where $m$ is the mass, and $a = \mathrm{d}\,\boldsymbol{v}/\mathrm{d}\,t$ the acceleration of the body with mass $m$. It is not possible to define force and mass independently [4]. $\rho$ is the mass density. To which problems are these definitions applicable? Are there other definitions? If so, how do they differ in their nature?

First of all, Newton's definition involves the inertial mass with unit kg [3, 4]. It is therefore implied that a discrete body with given mass $m$ is allowed to move in free space without mechanical interaction with other bodies. (In the theory of gases, the effect of atomic collisions can be ignored because atoms cannot be deformed; the collisions are "perfectly elastic". A car engine may be considered a free-body problem if the motions of its parts are thought to be friction-free.) Newton's force refers to a velocity potential [4] since work is done by positive or negative acceleration. Free-body problems are abundant in physics;

however, they are in the conservative class, from celestial mechanics to particle physics. Thus Newton's definition of a force does not apply to processes involving other masses, or to potentials that do not relate to motions of bodies in free space.

The thermodynamic mass is dimensionless and incompatible with Newton's definition of a force. In order to compare the thermodynamic or chemical potentials of two bodies consisting of different substances, say, K and Fe, their inertial mass is divided by the respective atomic weight, whereby the inertial properties are lost, and by Avogadro's number, resulting in a quantity measured in mol. In thermodynamics, quantities of atoms or molecules are compared with one another; their inertial properties are as irrelevant as heat is irrelevant in mechanics. Thermodynamic processes involve thermodynamic potentials, such as the internal energy $U$, the free energy terms, and others. A thermodynamic force, exerted by a system on its surrounding, is defined as

$$\boldsymbol{f} = \boldsymbol{e}_i \partial U / \partial x_i . \tag{2}$$

[1:52, 1:142]. The thermodynamic density is measured in mol per volume, $n/V = P/RT$, whereas the inertial mass density is irrelevant.

Finally, a Newtonian force cannot be used to describe a change of state, and a thermodynamic force does not refer to the velocity of bodies in free space. $\boldsymbol{f} = m\boldsymbol{a}$ is one single force vector only, it is not and cannot be a field force. The mechanical concepts are relevant to free-body motions only, they do not consider their internal state. Deformation-causing forces in solids surely are field forces since e. g. by displacing one atom relative to its surrounding in one particular direction, work is done not just in displacing the mass of the atom, but by changing the bond lengths in all directions. Field forces must be derived from a potential as in (2). It is not possible to transform an equation of motion into an equation of state, hence it is incorrect to use an equation of motion to describe a change of state. The Eq. (1) are obsolete at least since the invention of thermodynamics. The theory of stress $\sigma$ is solidly based on the equation of motion above [2, 3, 5]. Thus if a deformation is partitioned into a volume term $P \mathrm{d} V$ and the volume-neutral strain term $\sigma_{ij} \mathrm{d} \varepsilon_{ij}$, the equation

$$\mathrm{d} U = T \mathrm{d} S - P \mathrm{d} V - \sigma_{ij} \mathrm{d} \varepsilon_{ij} \tag{3}$$

is an impermissible mixture of references to the thermodynamic mass in the first and second term RHS, implying non-conservative processes, and to the inertial mass in the last term.

### 3. The Continuity Equation

The continuity equation, first written by Euler, is [3]

$$\frac{\mathrm{d} \rho}{\mathrm{d} t} + \rho \frac{\partial v_i}{\partial x_i} = 0, \tag{4}$$

where $\rho$, as above, is the inertial density. The continuity equation is a mass conservation law: if $\mathrm{d} \rho / \mathrm{d} t = 0$, the remaining equation is a Laplace equation. $\boldsymbol{v}$ are the velocities of all mass differentials within a region which is separated by a chosen closed interface from its surrounding. The equation states that within that region the paths of all mass differentials cancel, such that mass is conserved.

The unstated assumptions in the use of the equation are: **1.** that mass is the variable of interest; **2.** that the mass differentials can move freely past one another; **3.** that only radial motions of mass differentials need to be considered.

**1.** The assumption that mass is the variable of interest. If a process $f = f(a, b)$, and if it is only considered to be $f = f(a)$, $b$ is assumed constant. Vice versa for $b$. Replace $a$ with mass and $b$ with energy, and the point made above is proven. The continuity equation is designed to ensure that the paths of all mass differentials within the region cancel such that $\rho$ is constant and mass is conserved. However, the equation has the side effect that energy is conserved, too. If work is some linear or non-linear function of the paths of the mass differentials only, it follows that the work done in achieving the deformation cancels if the paths cancel; thus no net work is done. Possibly Euler intended the implication; he had been a student of Johann Bernoulli who discovered the energy conservation law of conservative mechanics (that the sum of the kinetic and the potential energy is invariant). Euler did not know yet that the energy of a system (the potential) can be a variable, in fact he did not even know the concept energy; it was introduced by Thomas Young in 1787, and only then the concepts of force and energy could be clearly distinguished.

**2.** The assumption that the mass differentials in the region can move freely past one another, especially free of bonds and friction. This is a principle of mechanics in general; without it, the process fails to be conservative, which is all Newtonian mechanics is about [4, 6]. Although this is hardly a correct assumption for elastic solids, it must be considered that the early workers were not thinking about elasticity. They were concerned with the flow of water; there the assumption regarding free motion appears to be more justified at first sight, except that viscous flow is today known to be an irreversible process. It was Tresca [7] who first "wished to apply the equations to solids which Cauchy had developed for water"; and he wanted to study plastic deformation, which is irreversible again.

**3.** The assumption that only normal motions of mass differentials need to be considered. This follows from the mathematical form of the divergence theorem which is, in the context of the terms used in the continuity equation,

$$\int \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}A = \int \nabla \cdot \boldsymbol{v} \, \mathrm{d}V = 0, \tag{5}$$

where $\partial v_i / \partial x_i$ in (4) equals $\nabla \cdot \boldsymbol{v}$ in (5). Thus, tangential motions are explicitly excluded from consideration: it is therefore implied that only normal motions of mass differentials contribute to the flux of interest. Since mass fluxes are the subject of the equation and Bernoulli's law is implied (see **2.**), and since mass is proportional to the energy of the system, the point is proven. The opposite – that tangential components contribute to the total work done – is precluded by the weight given on div $\sigma = 0$ for a volume-neutral deformation [2:101] because div $\sigma$ is meaningless if it is not implicitly proportional to the *total* work done. The identical conclusion follows from Cauchy's stress tensor (see below).

The consequences of the continuity equation are profound.

First, in choosing a founding stone for continuum mechanics, a mass conservation law was given preference where an energy conservation law is required since elastic deformation is a non-conservative energetic process. This choice of preference gives the entire body of continuum mechanics a conservative mathematical structure, i. e. it is understood as a Laplace problem whereas it is certainly a Poisson problem, as is all of continuum physics [8]; it cannot be anything but phenomenological. The point of interest in deformation theory is its energetics, not the mass distribution in space. By the nature of the process under discussion it is permissible to make the implicit assumption that mass is conserved, in the same way as it is always assumed in equilibrium thermodynamics that the system is closed with respect to mass.

Second, it must be considered that the mass differentials in a chosen region of solid cannot possibly move freely past one another, or else an elastic potential cannot build up. The continuity equation is only and exclusively applicable to infinitely slow equilibrium flow of a real gas where the assumptions (a) of free motion, (b) that non-radial motions can be disregarded, and (c) of zero net work do hold, hence where indeed an elastic potential does not build up. But this case is not the subject of interest in deformation theory. It is, in fact, impossible to define an elastic potential for a volume-neutral deformation in the continuum mechanics theory based on Euler; the equation of motion can also be given as div $\sigma = 0$ [2:101; 9:7-9], which is an explicit statement that no elastic potential exists, hence no net work is done: the divergence is a measure of the work done on a system [1:48-52]. Consider (3) above. The condition $\mathrm{d}\varepsilon_{ii} = 0$ (summation implied) is the definition of a volume-neutral deformation. Thus, the equilibrium condition in current understanding dictates that $\sigma_{ii}$ (being zero) is to be multiplied with a strain term, the trace of which $\varepsilon_{ii}$ must be zero as well, such that the product term $\sigma_{ij} \mathrm{d}\varepsilon_{ij}$ has no chance of being anything else but zero. If the process is reversible, $T \mathrm{d}S = 0$; if the deformation is volume-neutral, $P \mathrm{d}V = 0$; since both $\sigma_{ii} = 0$ and $\varepsilon_{ii} = 0$, it follows that $\sigma_{ii} \mathrm{d}\varepsilon_{ii} = 0$, hence $\mathrm{d}U = 0$. Indeed, Snedden and Berg [10:13-15] derive an expression for an elastic potential; however, since their theoretical approach is based on the theory of variations which develops the concept of virtual work [4], it is itself a conservative concept. The numerical value of their expression must again be zero.

Third, the normal displacements of the mass differentials in the region indeed must cancel, and inevitably the work connected with these motions must cancel as well if the deformation is volume-neutral, independent of the work function being used. The true error in the continuity equation (or rather, in the profound significance assigned to it) is that in using an equation similar or identical to the divergence theorem the assumption is implied that $\int \boldsymbol{f} \cdot \boldsymbol{n} \, \mathrm{d}A = \int \nabla \cdot \boldsymbol{f} \mathrm{d}V = \varphi$ represents the *total* divergence of the system, whether $\varphi = 0$ or not, i. e. it is implied that tangential forces do not contribute to the divergence, or to the work done. If the mass differentials could indeed move freely, there would be no contribution to the work done by shear forces or shear motions, and the assumption is correct. However, the validity of the divergence theorem is already restricted if the boundary of the region passes through points of mass [1:43]. Furthermore, in solids the mass differentials are bound in place by bonds in all directions. Hence if one mass differential is displaced in one direction, work will be done on the bonds in all directions. It is a simple experimental task to subject a body of, say, circular shape to tangential forces only, and still maintain the equilibrium condition that $\int \boldsymbol{f} \times \boldsymbol{r} \, \mathrm{d}A = 0$. The body will react (by expansion), hence work is done. The effect is known phenomenologically as dilatancy [11]. In continuum mechanics the question as to how much work is contributed to the total work done by the shear forces cannot be answered, it cannot even be asked.

Neither the inertial mass density $\rho$ nor the thermodynamic density $n/V$ is a state function by definition. If bonds exist within the volume of mass, it is possible to change the energetic state of a system through mechanical work without changing its volume – the standard case of an "incompressible" deformation – and the density remains unchanged. The thermodynamic density may be used as a state function with the understanding that all parts of the system are free to move past one another, such as a real gas. The condition does not hold for solids with internal bonds. In any case, it is not possible to write a meaningful equation of state using the inertial density $\rho$.

### 4. *The Equilibrium Conditions*

Two points shall be made in this section: **1.** that current continuum mechanics uses an inappropriate equilibrium condition; **2.** that the orthogonality of stress is not an equilibrium condition, but a boundary condition.

**1.** Types of equilibrium conditions. In his derivation of the stress tensor, Cauchy assumed a volume element in the shape of a tetrahedron. It is assumed that forces act upon the surface elements from outside; they are balanced by forces acting on the same surfaces from the inside. "According to Newton's third law, the mutual action of a pair of particles [within the tetrahedron] consists of two forces acting along the line interconnecting the particles, equal in magnitude and opposite in direction to one another. Therefore the resultant internal force is zero" [12:95]. The statement translates into the equilibrium condition $\partial\sigma_{ij}/\partial x_j = 0$ [9:7-9]. Umow [5] explicitly states that "the case of external forces acting on the parts of the body is not considered".

The equilibrium principle must be distinguished from the equilibrium condition. The principle is to be universally observed, whereas the chosen condition characterizes the process under discussion; choosing the wrong one will lead to a misconception. Newton's equilibrium conditions apply to the free motion of a discrete body in space. That is, whichever forces are observed, their origin is exterior to the body, and they are in balance with one another, so the body is at rest. Newton's equilibrium condition is explicitly not concerned with the internal state of the body; it cannot be considered.

The equilibrium condition above is a statement that the system of mass is in equilibrium with itself, i.e. it is unloaded. The relation between the interior and the exterior of the volume element, or between a system and its surrounding, is the subject of the thermodynamic equilibrium condition which mandates that the fluxes into and out of a region in space must balance,

$$\begin{aligned} &\text{in tensor form } \nabla^2 U_{\text{int}} + \nabla^2 U_{\text{ext}} = 0, \\ &\text{in vector form } \boldsymbol{f}_{\text{int}} + \boldsymbol{f}_{\text{ext}} = 0, \\ &\text{in scalar form } P_{\text{int}} + P_{\text{ext}} = 0 \end{aligned} \tag{6}$$

for isotropic conditions. Consider a solid in the unloaded state. Its volume is only determined by the strength of the internal bonds; the body is said to be in equilibrium with itself. In that case $\nabla^2 U_{\text{int}} = \nabla^2 U_{\text{ext}} = 0$, i. e. the Laplace equation is a valid condition – but only and exclusively in the unloaded state. The body, here taken as a thermodynamic system, is therefore in its zero potential state.

If the body is loaded, the externally applied forces will interact with the internal bonds by changing the bond lengths. Therefore the body develops an internal non-zero potential to return into its old configuration, its equilibrium material state. Thus the application of the external forces results in the development of non-zero internal forces that interact with the external ones such that they together form a new equilibrium state of the form above; the internal forces are therefore *evoked* forces. The quote by Eringen above says that along a loaded bond, the two parts on either side – volume differentials, at least two atoms – will exert equal forces with opposite sign upon each other, therefore equilibrium exists. This is not so; in the state of equilibrium the two forces would have zero magnitude. The question is not whether the forces pointing to the left and right within the volume element balance; since they are internal forces, the question is whether they are non-zero, and whether or not they are balanced by external forces acting on the system from outside. If imbalance exists, the body would undergo an unrestricted expansion, but not a motion in the Newtonian sense. Consider a one-dimensional model across a system:

$$\boldsymbol{a} \rightarrow | \leftarrow \boldsymbol{b} \cdot \boldsymbol{c} \rightarrow | \leftarrow \boldsymbol{d},$$

where the externally controlled forces $\boldsymbol{a}$ and $\boldsymbol{d}$ act from the outside on the surfaces (vertical bars) of a body centered between $\boldsymbol{b}$ and $\boldsymbol{c}$. The Newtonian equilibrium balances $\boldsymbol{a}$ and $\boldsymbol{d}$ because they are pointing left and right. However, since only balanced external forces can cause a change of state – unbalanced external forces would cause an external acceleration – $\boldsymbol{a}$ and $\boldsymbol{d}$ also point in the same direction in the sense that they point outside-in. $\boldsymbol{b}$ and $\boldsymbol{c}$ are evoked forces, they are mirror images of one another, acting inside-out. $\boldsymbol{b}$ and $\boldsymbol{c}$ are functions of the equilibrium of $\boldsymbol{a}$ and $\boldsymbol{d}$; a disequilibrium or gradient between $\boldsymbol{b}$ and $\boldsymbol{c}$ cannot exist. The thermodynamic equilibrium is that of the inside-pointing forces $\boldsymbol{a}$ and $\boldsymbol{d}$ vs. the outside-pointing forces $\boldsymbol{b}$ and $\boldsymbol{c}$. Newton's third law refers to Bernoulli's energy conservation law, not to the first law of thermodynamics. This is clearly not apparent in the quote above.

**2.** Orthogonality of stress. The Newtonian condition for rotational equilibrium is $\int \boldsymbol{f} \times \boldsymbol{r} \mathrm{d} A = 0$. $\boldsymbol{r}$ is a radius of a solid, extending from the center of mass $Q$ of a discrete body to its surface point $P$, $\boldsymbol{f}$ is the force acting on $P$, and $A$ is the surface of the discrete body, i. e. it is a closed surface. If $A$ is convex, the condition can also be given as $\int \boldsymbol{f} \times \boldsymbol{r} \mathrm{d}\,\theta$, i. e. $\boldsymbol{f}$ and $\boldsymbol{r}$ are then a function of direction relative to $Q$. It is then obvious that the shape of the body given by some shape function $f = \boldsymbol{r}(\theta)$ has a strong influence on the particular form of the equilibrium. The sum of dextral and sinistral rotating forces integrated over $\theta$ alone may be seriously lopsided, but the rotational momentum is still balanced if the properties of the vector field $\boldsymbol{r}$ vary inversely to those of $\boldsymbol{f}$, hence there is no reason to assume that the eigendirections of the field **f** alone must be orthogonal. There is only one minimal constraint: the eigendirections must not coincide, because then it is impossible to attain rotational equilibrium.

Apparently, Euler perceived water as a kinetic system of infinitesimally small bodies. The last paragraph above applies specifically to the equilibrium of one single discrete body. For a kinetic system of $n$ discrete bodies freely moving about in space there is no such thing as an overall equilibrium condition, be it normal or rotational, there is at best a zero flux across its boundaries. Yet the external (thermodynamic) equilibrium condition between system and surrounding was unknown to Euler. His orthogonality requirement looked like an equilibrium condition, but it never was one: whether or not a volume of gas or water is in a state of equilibrium, if subjected to an orthogonal configuration of forces, can be checked simply by clapping the hands together. (It would work for a cube or sphere of solid because of the bonds; but a volume of solid is no longer a kinetic system.) However, Euler's real error is someplace else. It seems that he started with his group of planes with orientation $\boldsymbol{n}$ (an unit vector) intersecting in the point $Q$; thus the point of action of all average forces per area is $Q$. This deprived him of the radius vector $\boldsymbol{r}$, so he replaced Newton's cross product $\boldsymbol{f} \times \boldsymbol{r}$ by $\boldsymbol{f} \times \boldsymbol{n}$, which is entirely conjectural.

In modern textbooks a reference point is assumed relative to which some point of interest is then given, in (1): $\boldsymbol{x}_0$ and $\boldsymbol{x}$. $\boldsymbol{x}$ is the location of Euler's group of planes. It is then argued that the cross product $\boldsymbol{f} \times (\boldsymbol{x} - \boldsymbol{x}_0)$ of all forces $\boldsymbol{f}$ acting on the point $\boldsymbol{x}$ which is contained in all planes $\boldsymbol{n}$, and the distance $\boldsymbol{x} - \boldsymbol{x}_0$ is the rotational momentum of the "part" $\boldsymbol{x}$ (1). From this description it is not at all clear whether $\boldsymbol{x} - \boldsymbol{x}_0$ is a radius of a solid, and $\boldsymbol{x}_0$ qualifies as the center of mass of a thermodynamic system or a discrete body, or a distance in free space between the origin $\boldsymbol{x}_0$ of some coordinate system, and $\boldsymbol{x}$ which is the location of a "part", i. e. one of $n$ parts of a kinetic system, akin to a real gas. In the first case the body or system would spin about itself, in the second it would spin about some external point. Surely the difference is not without physical significance, yet the textbooks are surprisingly vague here. However, it has never been claimed that $\boldsymbol{x} - \boldsymbol{x}_0$ vanishes during Cauchy's continuity approach, therefore it is invariant. Since a surface enveloping $\boldsymbol{x}_0$ and passing through $\boldsymbol{x}$ can have only one single orientation in $\boldsymbol{x}$, but $\boldsymbol{x}$ is the point of convergence of Cauchy's continuity approach and the location of Euler's group of planes, the distance

$\boldsymbol{x} - \boldsymbol{x}_0$ is indeed just the purely geometric location vector of point $\boldsymbol{x}$ with respect to some coordinate system in Euclidean space, but not a physically relevant term such as Newton's radius $\boldsymbol{r}$ which, after all, can act as a lever; $\boldsymbol{x}$ cannot be used in the statement of the rotational momentum. It is done anyway (1).

The cross products $\boldsymbol{f} \times \boldsymbol{n}$ and $\boldsymbol{f} \times (\boldsymbol{x} - \boldsymbol{x}_0)$ where $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{n})$ have the same birth defect: both $|\boldsymbol{n}|$ and $|\boldsymbol{x} - \boldsymbol{x}_0|$ are invariant, hence if they are taken in lieu of $\boldsymbol{r}$ in the cross product with all possible $\boldsymbol{f}$ acting on $\boldsymbol{x}$, their magnitude invariance implies a specific shape, that of a sphere. A spherical body of solid is in rotational equilibrium with all external force fields that have orthorhombic or higher symmetry; for lower symmetries the geometric properties of the field and the system shape must cancel one another. Therefore, the assertion by both Euler and Cauchy that stress is orthogonal by nature, is based on Euler's incorrect reinterpretation of Newton's definition of the rotational momentum, and amounts to the implicit assumption of a spherical system shape. The orthogonality condition is indeed equivalent to an equilibrium condition – for Laplace problems. Deformation of solids is not one of them.

The lack of generality in the Euler-Cauchy approach to stress has the effect of a hidden boundary condition. Current continuum mechanics often appears to deliver useful predictions in those cases where the experimental boundary conditions are sufficiently highly symmetric to be in accordance with the unrecognized assumption of a spherical shape of the system. It fails systematically for all problems involving simple shear.

## 5. *What is Stress?*

Stress is said to be "a system of forces" [2], but it is not a force field. Euler saw the necessity to describe vectors with a direction as a function of another direction. Adapting Newton's definition of pressure ($P = |\boldsymbol{f}|/A$) to his purposes, he conceived the cut model: a group of planes pass through the point $Q$ in space for which the state of stress is to be described; the orientation of a plane is given by a unit vector emanating from $Q$ in the plane, and the force vectors vary as a function of orientation of the plane.

This concept must be seen in its historical context. Euler did not know vector fields yet. He published in 1776 and passed on in 1783. Lagrange derived the first vector field in the following year. Pressure became recognized as a thermodynamic state function – and thus a scalar by definition – in the early 19th century. Looking around in physics, this "system of forces" is unique, and used exclusively in continuum mechanics because Euler's plane orientation vector $\boldsymbol{n}$ is not identical to $\boldsymbol{n}$ in the LHS of the divergence theorem (5): although they both are surface-normal unit vectors, Gauss's surface is closed whereas Euler's planes are free. Work is not of much concern; the basis of the understanding are the equation of motion and the equation of continuity, whence it is concluded that for a volume-neutral deformation $\operatorname{div} \sigma = 0$ [2:100-101, his Eqs. (3) and (10) ignoring body forces]. This statement is the most obvious evidence that the Euler-Cauchy theory of stress is wrong.

In modern terms, stress is the elastic potential, it is the work done during loading (a scalar), and its spatial properties are represented by a force vector field. An unloaded body is in its zero potential state; during loading, work is done by external forces upon the system, and that work is stored as potential to reverse the change of the energetic state. Since work has been done to the effect that a change of state occurred, the divergence of stress cannot be zero. It is telling that the term "elastic potential" is almost completely missing in most standard continuum mechanics textbooks.

One has to be careful when reading the literature on theoretical continuum mechanics because some terms have a meaning very different from that in other disciplines of physics. Throughout thermodynamics, the all-pervading question is the equilibrium between system and surrounding. The system is usually a subvolume of mass within a larger continuum of mass; forces acting upon the system are understood to be *external forces*, and forces exerted by the system upon the surrounding are then *internal forces*. In contrast, continuum mechanics does not know the concept of a system of mass. Here, internal forces are forces acting *across a free surface inside a solid*, and they may be balanced by each other in the way intended by Eringen [12:95], i. e. the understanding of the equilibrium presupposes the validity of the Cauchy lemma (below). It is not implied that the body or a volume element of it is a thermodynamic system, nor is the interface across which internal forces inside a solid act, thought to be the boundary of a thermodynamic system; if so, it would necessarily be a *closed surface*, with very different mathematical implications. The difference between inside and outside, system and surrounding in thermodynamics (in accordance with the divergence theorem) does not exist in continuum mechanics: there, external forces act upon the surface of the body as a whole, and the internal forces are thought to be their continuation into the interior of the body. From the entire theoretical structure of continuum mechanics, forces exerted by a subvolume upon its surrounding and forces exerted by the surrounding upon the system cannot be recognized as different in origin: continuum mechanics does not ask for the source of the forces (the potential), hence it does not find any. The concept of a thermodynamic system and the Cauchy stress are incompatible with one another.

## 6. *Hooke's Law*

The mathematical structure of Hooke's law greatly influenced the thinking about deformation. However, **1.** Hooke's law is a phenomenological law, **2.** it is a seriously flawed misconception to the effect that it has the wrong unit.

**1.** The phenomenological character is evident since the "linear" stress-strain relation implied by it is long known to be something of an idealization; whether it is an idealization towards a physical ideal, in analogy to the ideal gas, or towards the observer's personal prejudices remains yet to be openly discussed. Instead, one speaks of a Hookean body if the material behaves according to Hooke's law; other definitions of bodies associated with certain types of behavior are also known. The circular reasoning should be obvious: because it is a Hookean body, Hooke's law is valid; in other cases we take other laws. In many books on the basics of deformation theory one can find caveats to the effect that the theory is applicable only to small strains; in other words, the theory is known to be wrong, but at small strains the error does not matter.

Commonly it is assumed that the (Hookean or other) body is "incompressible", meaning that it can be deformed at will, but its volume and mass density remains constant. (The ideal material ratios, e. g., Poisson's ratio or the Lamé constants, are contingent on volume constancy.) Once this line of thought has been accepted, it is hard to ask why on earth the volume indeed does remain constant; and worse, an observed change of volume will then be perceived as an anomaly. The mistake in this line of thinking is subtle. The *material* is thus said to be incompressible whereas it is the *process* that is volume-constant.

A proper approach should not start with implicit boundary conditions that please the expectations, but one should arrive at results such as volume constancy by deducing them from a proper approach. If a material does not behave according to expectations, for example because it expands, it is called dilatant. Thus if the material is supposed to be dilatant, the phenomenon of dilatancy can no longer be perceived as a function of the experimental setup. The concept of a dilatant material is the phenomenological flipside to the incompressible body. Fact is, dilatancy is regularly observed in elastic simple shear [11], i. e. as a function of the boundary conditions and independent of the material.

If a bar is stretched in $x$, it is observed that it will attenuate in $y$ and $z$. It could be concluded: provided that the volume is constant, and if the bar is stretched in $x$, it must shorten by this and this amount in $y$ and $z$. The result is Poisson's ratio which is again phenomenological: it explains an observation with itself. The physical line of thought should be: if work is done by the surrounding on the bar in $x$, changes in $y$ and $z$ are nevertheless observed, therefore work is done by the system on the surrounding in these directions. How come? (The reason is the principle of least work. A stretching in $x$ without attenuation in $y$ and $z$ would result in a volume change, which is a much larger change of state per stretching increment.) This approach will lead towards far better understanding of deformation than Poisson's ratio or the Lamé constants, and, by the way, the presupposition of volume invariance is no longer necessary.

**2.** Hooke wrote: *ut tensio sic vis*. The question is what was meant by *vis* at a time when force and energy were not yet conceptually separated. *Vis* is commonly translated as force, but Bernoulli's *vis viva* and *vis mortua* are today's kinetic and potential energies of classical mechanics (the latter terms were introduced by Thomas Young in 1787). – Newton's definition of a force is $\boldsymbol{f} = m\boldsymbol{a}$. If $Z$ is the potential energy of conservative mechanics, $\partial Z/\partial x_i = f_i$ [4]. The force derived from a thermodynamic potential $U$ is $f_i = \partial U/\partial x_i$ [1:152]. Hooke's law in its simplest form is $f = c\Delta l/l_0 = c\varepsilon$, where $c$ is the spring constant or Young's modulus, $l_0$ the length of an unloaded bar or spring, and $\varepsilon$ the strain (whereby *tensio* is thus translated as lengthening). Hooke's law is therefore some kind of definition of a force, called a material law.

But why a *force*? Newton's force refers to a velocity potential [4]. The two others are definitions of force fields derived from energetic potentials, i. e. energy fluxes as a function of direction. Furthermore, Newton's work is force times distance $w = \boldsymbol{f} \cdot \boldsymbol{x}$, and the thermodynamic definition of mechanical work is $\mathrm{d}w = \mathrm{d}U = -P\mathrm{d}V$ (if $\mathrm{d}q = 0$, $\mathrm{d}w$ is an exact differential). A force is the cause; the displacement or the volume change is the effect. Work is the energy spent in achieving that effect. A change of length is an effect, so Hooke's law evidently is a first attempt to quantify the *work* done in stretching the spring, and it should have the unit Joule. The missing energetic term must then be the force which is applied. Given that a change of state is involved, and that the work function for such a change can only be logarithmic (see below), a preliminary choice for a one-dimensional law should be $\mathrm{d}U = f\mathrm{d}l$, in analogy to $P\mathrm{d}V$.

We are left to find out what is meant by *tensio*. If Newton's definition of work gives any hints, $w$ is the product of the applied force and the displacement caused by it. Both Newtonian work and thermodynamic work are path-independent. But whereas Newtonian work is path-independent in Euclidean space – the state of the entire free-space system of bodies is, after all, invariant in classical mechanics – the thermodynamic work is path-independent in $PV$-space, which is the energy space: starting from some reference state, a particular final state always requires the same amount of work, regardless of the path taken in that space, provided the process is reversible.

Newton's work definition clearly indicates that the *displacement* is the quantity of interest. In current continuum mechanics the fundamental relation of importance is that of stress to *strain*, i. e. the mere change of shape, due to Lagrange's interpretation of Hooke. The requirement of path-independence indeed seems to support the classical interpretation since strain is path-independent whereas the displacement field appears to describe a path – in Euclidean space, but not necessarily in energy space. But a finite displacement can also be reached through an infinite number of displacement histories, and the requirement that the energetics of the final state be independent of the deformation history may therefore also be made with regard to the final state of displacement, rather than the state of strain. Whether strain or displacement is the energetically relevant term is best decided by looking at the energy used because the work done defines the state which the system acquires. The difference is only apparent if the properties of strain and displacement differ; whereas strain is by definition an orthogonal tensor, the displacement field is a vector field that may or may not be orthogonal. If it can be shown that the energetics of strain are independent of the properties of the finite displacement field, the significance of the classical stress-strain relation is supported. If it can be shown that identical states of strain require different amounts of work as a function of the finite displacement field chosen, the physical relevance of the stress-strain relation is refuted.

The experimental record is clear. It is long known that elastic simple shear (non-orthogonal displacement) requires more energy than pure shear (orthogonal) in order to achieve the same total strain [11]. Even in the plastic field the energetics of simple and pure shear vary systematically and independent of the material [13; pers. comm. 1989], although here the energetics of simple shear are consistently (and substantially) lower than those of pure shear. All these differences should not exist if the stress-strain relation bore any significance to the energetics of deformation, and they cannot be predicted by the Euler-Cauchy approach.

## *7. Cauchy's Continuity Approach*

Cauchy ventured to put the cut model on a more solid mathematical basis. He assumed a volume element within a larger volume of solid or fluid; forces act upon its surfaces. He then made the assumption that the ratio $|\boldsymbol{f}|/A$ should be scale-independent. By letting the faces of the volume element approach a point of convergence by means of a limit operation with respect to volume (similar to those in differential calculus), he then assumed that the forces per area should reach a finite value, such that in the moment of convergence three goals are achieved: first, that the state of stress be described through the Cauchy lemma [3]

$$\boldsymbol{f}_{-x} = -\boldsymbol{f}_x, \tag{7}$$

second, that the continuum of points is reached, and third, that the shape of the original volume element vanishes. All three assumptions imply that the volume element vanishes identically.

The hidden assumptions in this line of thought are: that the continuum of points is the key to understanding of deformation; that the limit of the limit operation exists; that the force magnitude $|\boldsymbol{f}|$ is independent of scale = the magnitude of the volume element; that the equilibrium conditions are unaffected by the limit operation, and that the shape of the volume element vanishes. All these assumptions are interrelated, and they all violate the same principle, that of proportionality of mass and potential which is the fundamental existence theorem of the theory of potentials.

Behind the concept of a continuum of points is the idea of the point source in potential theory. A *point source* is a body of finite extension and mass. The mass of the body is thought to be concentrated in one point such that its mass is preserved, but its volume is considered to be zero for simplicity. This concept causes no problems in many cases. The condition for considering bodies as point sources (e. g., in gravity) is that the bodies in a system (such as a planetary system) are discrete bodies (that it is possible to envelop them with a surface that does not pass through mass), that the bodies do not interact mechanically, and that the distance between the point of interest outside a body and the body itself is so large that its shape no longer has an effect.

The principle that is maintained in the point source concept is the proportionality of potential and mass. Consider a planet in free space causing a gravitational force field in the region around it. We consider the flux out of a region that contains the source ($\varphi \neq 0$). The potential is then zero at infinity (the zero potential distance is infinite; [1:53, 15]); at any smaller scale there will be a flux through the boundaries of the region. The flux is always proportional to the magnitude of the source and to the mass in the region, but independent of the volume of the region which is empty except for the relatively small part of it occupied by the planet. Thus it follows that, since the region contains mass, in

$$\int \boldsymbol{f}\cdot\boldsymbol{n}\,\mathrm{d}A = \int \nabla\cdot\boldsymbol{f}\,\mathrm{d}V = \varphi = \text{const} \tag{8}$$

the RHS integral is independent of the limits of integration of $V$. Thus $\int \boldsymbol{f}\cdot\boldsymbol{n}\,\mathrm{d}A$ must also be constant. Thus $|\boldsymbol{f}| \propto A^{-1}$, or, in terms of the radius $r$ of the region, $|\boldsymbol{f}| \propto r^{-2}$. This is Newton's law of gravitation.

However, the Cauchy continuity approach considers a subvolume in a larger region continuously filled with mass. Instead of a point source we have a *distributed source* [1, Chapt. 2.8 and Chapt. 6.3, leading to the Poisson Eqn.]. By the principle of proportionality of mass and potential – and, for simplicity, assuming homogeneous mass distribution at the scale considered – it follows that the charge is proportional to the volume, $\varphi \propto V$; it follows that $\nabla\cdot\boldsymbol{f}$ is a constant. Since the RHS is proportional to $V \propto r^3$, for the equality of RHS and LHS to hold it follows that, since $A \propto x^2$, it is concluded that $|\boldsymbol{f}| \propto x = r$, or [1:19, 14]

$$\frac{|\boldsymbol{f}|}{|\boldsymbol{r}|} = \text{const.} \tag{9}$$

Thus within a larger continuum of mass, the magnitude of the forces exerted by the system upon the surrounding is a function of scale. They must vanish together with $r$.

The result is in perfect agreement with the fundamental existence theorem of potential theory: let $\varphi$ be some function $f$ of the point $Q$ within the region $V$ in space. The integral $\int f(Q)\mathrm{d}V = \varphi$ is convergent only if it vanishes with $r$ [1:147]. The Cauchy lemma above assumes that a finite value is reached as $V$ vanishes; however, the limit does not exist [14].

## 8. *The Notation of Planes in Space*

A vector has a magnitude and a direction. The concept of vectors can be applied only if a minimum set of requirements are met to ensure that the correlation of vectors and objects is unique – that no two objects can be assigned the same notation, that no two notations apply to the same object, and that no object is without notation – and to ensure that algebraic minimum logic is observed. A full set of conditions which together comprise the definition of a vector space, can be found in [4:5, 16:131]. Of interest here are the conditions that any object $\boldsymbol{u}$ and its negative $-\boldsymbol{u}$ must be two different objects; that the object $\boldsymbol{0}$ exists such that $\boldsymbol{u} - \boldsymbol{u} = \boldsymbol{0}$; and that the scalar multiplication $k\boldsymbol{u}$ is a meaningful operation such that the magnitude of $k\boldsymbol{u}$ is the product of the scalars $k$ and $u = |\boldsymbol{u}|$, and if $k = 0$, $k\boldsymbol{u} = \boldsymbol{0}$. The null vector $\boldsymbol{0}$ is said to be a zero vector quantity which has neither magnitude nor direction.

The commonly used notation for planes in space is known as Hesse notation: starting from a reference point $Q$, any point $P$ with position vector $\boldsymbol{p} = Q \rightarrow P$ indicates a plane which contains the point $P$ such that it is perpendicular to $\boldsymbol{p}$. This convention serves to describe all planes in space except those that pass through $Q$ because for these planes the position vector is the null vector. That singularity is common to all coordinate systems. The Hesse notation satisfies the requirements for vector spaces.

Euler's cut model consists of a group of planes passing through a point $Q$. Their orientation is given by a vector $\boldsymbol{n}$ emanating from $Q$; $\boldsymbol{n}$ is by convention a unit vector. This notation serves to describe planes passing through $Q$ exclusively; other planes in space cannot be described. Some sources therefore introduce an auxiliary term [$\varepsilon$ in 3:176] such that the vector $\varepsilon\boldsymbol{n}$ indicates other planes with the orientation of $\boldsymbol{n}$, and $\varepsilon$ indicating the minimum distance of the plane to $Q$. This is identical to the Hesse notation except that if $\varepsilon = 0$, $\varepsilon\boldsymbol{n} = \boldsymbol{n} \neq \mathbf{0}$.

The two notations cannot be transformed into one another, thus they are not to be used simultaneously. Euler's notation does not satisfy the minimum requirements for vector spaces. If Gurtin's $\varepsilon = 0$, the resulting vector $\varepsilon\boldsymbol{n}$ must be a null vector $\mathbf{0}$. However, the object $\mathbf{0}$ does not exist in Euler's notation; the notations $\boldsymbol{n}$ and $-\boldsymbol{n}$ describe the identical object, and the operation $\boldsymbol{n} - \boldsymbol{n}$ is meaningless.

### *9. The Role of Shape in Mechanics*

In discrete mechanics the shape of a body is of paramount importance for the equilibrium conditions because of the rotational momentum $\boldsymbol{f} \times \boldsymbol{r}$. Depending on the shape of the body, the particular form of the equilibrium may differ vastly from others. The conclusion drawn in the Euler-Cauchy theory is that shape is irrelevant in continuum mechanics since there is no discernible shape to be observed. In the process of Cauchy's continuity approach the shape is thought to vanish identically.

Consider a body with given shape. Taking its center of gravity as the origin for a coordinate set, the shape is defined as some function of $\boldsymbol{r}(\theta)$, and the forces acting on the body may also be written as $\boldsymbol{f}(\boldsymbol{r}, \theta)$, where the point indicated by $\boldsymbol{r}$ is the point of action of $\boldsymbol{f}$. The condition of rotational equilibrium is then $\int \boldsymbol{f} \times \boldsymbol{r}\mathrm{d}\theta = 0$, according to Newton. (Letting $\boldsymbol{r}$ vanish without changing $\boldsymbol{f}$ would result in a trivial solution, and it would make the ratio $|\boldsymbol{f}|/|\boldsymbol{r}|$ arbitrary.) If a limit operation with respect to $\boldsymbol{r}$ is performed, a particular cross product $|\boldsymbol{f} \times \boldsymbol{r}|$ will naturally approach zero faster than $|\boldsymbol{r}|$ alone, but so does $\boldsymbol{f} \cdot \boldsymbol{r}$, and $\left(|\boldsymbol{f} \times \boldsymbol{r}|^2 + |\boldsymbol{f} \cdot \boldsymbol{r}|^2\right)^{1/2} = |\boldsymbol{f}||\boldsymbol{r}|$ is always proportional to both $\boldsymbol{f}$ and $\boldsymbol{r}$. Since $|\boldsymbol{f}| \propto |\boldsymbol{r}|$ (9), the ratio $|\boldsymbol{f} \times \boldsymbol{r}|/\boldsymbol{f} \cdot \boldsymbol{r}$ is scale-independent; the rotational momentum cannot vanish, so the shape continues to exist as long as $r \neq 0$.

In the above example, $\boldsymbol{r}(\theta)$ marks a point on the interface between a body of solid $V$ and its surrounding. That is, mass and potential of the body in the direction $\theta$ extend up to this point in the unloaded state, and no further. The surrounding may be free space or the continuum of mass of which $V$ is a subregion. $\boldsymbol{r}$ can be changed due to deformation, i. e. in the loaded state. $|\boldsymbol{r}_0|$ is therefore the *zero potential distance* $s_0$ which is an intrinsic geometric property of any potential problem [1:53]. It is a characteristic distance in Euclidean space which must be determined depending on the nature of the problem; $s$ may have infinite or finite length, but it cannot be zero. If $s$ is finite – by nature or by choice – it is by convention set to unit magnitude. A change of $s_0$ implies that a non-zero potential state is considered. Cauchy's limit operation is in effect an attempt to let that distance vanish; this is incompatible with the principles of potential theory and, in fact, with Hooke's law which indeed does contain the zero potential distance – the length of the spring.

In a continuum of mass, there are no easily discernible boundary conditions for the length of the zero potential distance. This is not to imply that $s$ is zero; rather it is to be assumed to have unit magnitude [1:63] unless other constraints are found. The surface defined by all $\boldsymbol{r}_0$ with magnitude $s_0$ is then the surface of the thermodynamic system which in the simplest case has spherical shape. The point will be discussed again in another paper [17].

## The Theory of Potentials

### *1. Properties of the Divergence Theorem*

The divergence theorem [1, Chapt. 4 to 6; 15]

$$\int \boldsymbol{f} \cdot \boldsymbol{n}\,\mathrm{d}A = \int f(Q)\mathrm{d}V = \int \nabla \cdot \boldsymbol{f}\,\mathrm{d}V = \varphi(Q) \quad (10)$$

considers fluxes crossing the closed boundary $A$ of a chosen region $V$ in Euclidean space. Gauss's theorem states that the flux crossing the boundaries of $V$ equals the divergence of the system, integrated over its volume. $\varphi$ is the charge of $V$ as a function of location $Q$ in space. If $\boldsymbol{r}$ is the position vector of a point $P$ relative to $Q$, $\int \nabla \cdot \boldsymbol{f}\mathrm{d}\boldsymbol{r} = \boldsymbol{f}$ is the force at $P$ as a function of the potential at $Q$. In terms of linear algebra, if $\nabla^2 U = \nabla \cdot \boldsymbol{f} = \mathbf{F}$, $\mathbf{F}\boldsymbol{r} = \boldsymbol{f}$ where $\nabla \cdot \boldsymbol{f} = \mathbf{F}$

is the field property tensor. It follows that $|\boldsymbol{f}| = 0$ if $\boldsymbol{r}$ is a zero vector [1:122, 135].

The physical interpretation of the divergence depends on the nature of fluxes. If mass fluxes are observed, the condition $\varphi = 0$ is an excellent mass conservation law. If gravity or electric charges are considered, the divergence is the source density, the attraction of the gravitational or electric potential per mass, and $\varphi$ is the gravitational or electric charge of the system under consideration with given magnitude. If the energetic state of a system is discussed, the divergence is a measure of the work done on the system per mass [1:48-52], and $\varphi = 0$ implies that the energetic state is invariant. All conservative processes must fulfil that condition; their energy conservation law is Bernoulli's law, $U_{\text{kin}} + U_{\text{pot}} = \text{const}$, which translates into the Laplace equation $\nabla^2 U = 0$. The energy conservation law for non-conservative processes is the first law of thermodynamics, $\mathrm{d}U = \mathrm{d}w + \mathrm{d}q$ (Poisson equation: $\nabla^2 U = \varphi \neq 0$). Again two groups must be distinguished: the reversible and the irreversible class. General solutions for the Poisson equation exist only for reversible processes, e. g. the Helmholtz equation.

Gauss's theorem is an explicit statement of the assumption that only normal forces contribute to the total divergence. This assumption cannot be verified by considering mathematics, but only by inspecting the nature of the physical problem under discussion. If all parts of a system can be moved freely past one another, the assumption holds, and the trace of the tensor $\nabla^2 U = \nabla \cdot \boldsymbol{f}$ is a useful term. With regard to forces causing elastic deformation of a solid body, it is self-evident that shear forces do work which is not included in $\nabla \cdot \boldsymbol{f}$, hence the above equation needs to be amended, with the consequence that simple tensor mathematics becomes insufficient for the proper representation of the process. This subject will be discussed in a subsequent paper.

Usually precaution must be taken that the region $V$ is convex and simply connected, and that its boundary $A$ is free of cusps. Furthermore, in the form given above, $A$ is a closed surface that completely envelops $Q$. $\boldsymbol{n}$ is commonly understood to be a surface-normal unit vector. However, the first condition makes it possible to understand the fluxes not as a function of a particular point on $A$, but as a function of a direction $\boldsymbol{n}$ relative to $Q$ since for every direction there is one and only one surface point $P$. If the orientation of the surface-normal vector at $P$ differs from that of the direction vector $\boldsymbol{r}(P)/|\boldsymbol{r}|$, the relation must still be a transformation the physical effect of which cancels. Thus the relation of surface orientation to orientation of flux at $P$ is not as important than it might look at first.

This observation puts the significance of the orientation of the surface in Euler's cut model in perspective. Furthermore, since $V$ is a non-zero region, the radius exists, and Newton's definitions of the normal and rotational momentum, $\boldsymbol{f} \cdot \boldsymbol{r}$ and $\boldsymbol{f} \times \boldsymbol{r}$, can be applied. Euler's cut model does not know a radius $\boldsymbol{r} = Q \rightarrow P$; the attempt to use the surface orientation vector in the equilibrium conditions, a common practice in continuum mechanics [2:101 ff.], is a violation of Newton's definitions. Finally, since Euler believed stress to be a form of pressure, he adopted Newton's definition "force per unit area". However, the more fundamental definition is the one provided by thermodynamics, $P = \partial U/\partial V$, or $P = U/V$ in integrated form which is the energy density. The cut model represents a group of free surfaces intersecting in $Q$; to such a group of planes the divergence theorem does not apply because the model fails to recognize point sources at the point of intersection. Consider a point source of heat located at $Q$, enveloped by $A$. The divergence theorem indicates that $\varphi \neq 0$, hence there is a non-zero flux of heat out of the system; if equilibrium exists, an independent external flux of similar magnitude must exist such that $\varphi_{\text{syst}} + \varphi_{\text{surr}} = 0$. Consider the free planes $A$ of the cut model intersecting at $Q$: the heat source is within the planes. Fluxes to either side of any plane have equal magnitude and opposite sign, hence the conclusion that $\varphi = 0$ appears to be permissible; this is incorrect.

The integral in the divergence theorem is convergent if it vanishes with the maximum chord of $V$ [1:147]. The limit cannot reach a finite value as is assumed in the Cauchy lemma [14].

## 2. *Definition of the Thermodynamic System*

The principle on which the theory of potentials is based, is the proportionality of mass $m$ and potential $U$ in a given state. If $\mathrm{d}m$ is a mass differential and $\mathrm{d}V$ is its volume, $\mathrm{d}m/\mathrm{d}V = \rho$, the density of the inertial mass, and $\int \rho \mathrm{d}V = m$ is the inertial mass of the system, the size of which is defined by the limits of integration.

With regard to the exterior of the system, the potential within it can be regarded to be located at the system's center of mass [1:7]. With regard to the interior, it is not possible to dissolve the distributed mass into a continuum of points with point sources (which was the intent of the Cauchy continuity approach); since mass is a variable if the region considered vanishes, the potential must vanish, too. A region continuously filled with mass is known as a *distributed source* [1, Chapt. 2.8 and Chapt. 6.3]; the classical example is that of gravity within planets. Given homogeneous mass distribution, the gravitational force exerted by a subregion of mass within a planet is proportional to its radius $\boldsymbol{f} \propto \boldsymbol{r}$ [1:19]. It is in the nature of gravity that the gravitational potential per unit mass is invariant; electrical, or thermodynamic potentials may vary.

The thermodynamic density is $\mathrm{d}n/\mathrm{d}V$, hence $\int \mathrm{d}n/\mathrm{d}V\,\mathrm{d}V = n$ where $n$ is the mass measured in mol. Therefore the system has a spatial extension and a shape. The limits of integration may be chosen to fit the desired purposes, thus arriving at the potential per mol, per unit volume, or per unit radius, times a proportionality constant. As above, the thermodynamic potential is proportional to mass, but a system may be considered to be in its zero potential state at some standard state which is subject to convention. It follows from the work function that thermodynamic potentials are logarithmic [1:53, 1:63], thus $s_0$ is finite; the distance term that is characteristic of a thermodynamic system in its standard state is the radius which is therefore assigned unit magnitude, $r_0 = 1$. Unless there is evidence to the contrary, it is safe to assume that the shape of the system is that of a sphere such that the surface/volume ratio is at its minimum.

An energetic differential can be imagined in two ways: if a limit is taken with respect to mass $n$ which is proportional to its potential $U$, the mass differential $\mathrm{d}\,\mathrm{n} \propto \mathrm{d}U$. At constant state, a change of $U$ at constant conditions is therefore equivalent to a change of mass; hence the definition of pressure may be given as $P = \partial U/\partial V$ or in integrated form, $P = U/V$, but note that mass is a variable in the integration, and measured in terms of $V$. An entirely different meaning of $\mathrm{d}U$ is implied if the potential $U$ of a given finite, constant amount of mass is changed by an infinitesimal amount $\mathrm{d}U$, resulting in a change of state. In the first case the principle of conservation of mass and energy are equivalent; in the second case they are not. Cauchy's limit operation known as the continuity approach considers the $\mathrm{d}U$ of the first kind, following the equation of motion in which the variable of choice is the inertial mass. But because elastic deformation is a change of state, the latter view must be taken. Conservation of mass and energy must be considered separately if both mass and energy are independent variables. However, the standard assumption in equilibrium thermodynamics that a system be closed with respect to mass, greatly simplifies the problem.

### 3. *Thermodynamic Approach in Vector Form*

Any change of state must be described by means of a state function. Here, isotropic loading of an ideal gas is explained in terms of vector fields instead of scalars in simplified form. $PV = nRT$ is assumed to be the correct state function for the material. Let the equation of state be reduced to Boyle's law $PV = c$. The work done due to a change of volume is found by differentiating,

$$P\,\mathrm{d}V + V\,\mathrm{d}P = 0, \quad \mathrm{d}P = -P\frac{\mathrm{d}V}{V}, \tag{11}$$

integrate and sort terms,

$$\Delta P = -P_0 \ln \frac{V_1}{V_0}, \quad \frac{\Delta P}{P_0} = -\ln \frac{V_1}{V_0}. \tag{12}$$

The work done is found from $\mathrm{d}w = -P\,\mathrm{d}V$ by substituting $c/V$ for $P$,

$$\int \mathrm{d}w = -c \int \frac{\mathrm{d}V}{V} = -c \ln \frac{V_1}{V_0}. \tag{13}$$

Since $P$ and $V$ are scalars, Boyle's law is isotropic. It may be understood not only as the product of two state functions, but also as the product of two vector fields, a radius field $\boldsymbol{r}$ and a force field $\boldsymbol{f}$, both of which are radial. If an area term is cancelled, the state function is transformed

$$PV =\rightarrow \frac{f}{A} rA =\rightarrow fr = \mathrm{const} \tag{14}$$

without changing the nature of Boyle's law (the unit is still the Joule). Thus, $P\,\mathrm{d}V$-work is equivalent to $f\,\mathrm{d}r$-work, and $U = PV$ is equivalent to $U = fr = |\boldsymbol{f}||\boldsymbol{r}| = \left(|\boldsymbol{f} \times \boldsymbol{r}|^2 + |\boldsymbol{f} \cdot \boldsymbol{r}|^2\right)^{1/2}$, normalized per unit mass times a proportionality constant to fit the condition $r = 1$.

A vector field is derived from a scalar quantity by differentiation with respect to the coordinates [1:52, 1:152]. A force field is derived from a potential $U$ which is characteristic for the energetic state at a given point $Q$ in space where the field is rooted. The directional flux $f_i = \partial U/\partial x_i$ is the force field; its spatial properties are given by the flux field property tensor $\mathbf{F} = \partial^2 U/\partial x_i \partial x_j = \partial f_i/\partial x_j$. For a radial field, $\mathbf{F} = c_1 \mathbf{I}$. The force $\boldsymbol{f}$ as a function of $Q$ is given by

$$\boldsymbol{f} = \mathbf{F}\boldsymbol{r} = \sum_i \int_{r_j} \frac{\partial f_i}{\partial x_j} \, \mathrm{d} x_j, \tag{15}$$

where $\boldsymbol{r}$ is the position vector of the point of action of the force $\boldsymbol{f}$ on the surface of the system relative to $Q$. By the same line of argument as above one can start with $fr = c$ and arrive at

$$\frac{\Delta f}{f_0} = -\ln \frac{r_1}{r_0} \tag{16}$$

and

$$\int \mathrm{d} w = -c \int \frac{\mathrm{d} r}{r} = -c \ln \frac{r_1}{r_0}. \tag{17}$$

In the present model case all forces are normal, thus $fr = \text{const}$ is only the dot product of $\boldsymbol{f}$ and $\boldsymbol{r}$ which is a scalar. Integrating the result over the surface of the system will yield the same result as above, times a proportionality constant. In this example, $\boldsymbol{f}$ may be either the outward-directed field $\boldsymbol{f}_{\text{int}}$ exerted by the system at the surrounding, or the inward-directed field $\boldsymbol{f}_{\text{ext}}$ exerted by the surrounding at the system. Since a system subjected to an external load interacts with its surrounding, $P_{\text{ext}} = -P_{\text{int}}$ translates into $\boldsymbol{f}_{\text{ext}} = -\boldsymbol{f}_{\text{int}}$; both equations are forms of the equilibrium condition of system and surrounding

$$\operatorname{div} \boldsymbol{f}_{\text{ext}} + \operatorname{div} \boldsymbol{f}_{\text{int}} = 0. \tag{18}$$

In the above example, where an ideal gas was used, all forces are normal forces since the state of loading was said to be isotropic, and shear forces do not exist.

## Conclusion

In many years of discussion I have met numerous people who confessed to have their doubts with regard to the present continuum mechanics theory. Typical was the reaction of a researcher who freely conceded that the understanding of deformation is insufficient, but he expected the solution somehow to come from an insight of an entirely new quality. Then he wiped the divergence theorem off the blackboard, it meant nothing to him, and explained to me the basics which evidently I had gotten wrong, starting with the equation of motion. The number of people who did not understand, or purported not to understand the difference between the energy conservation law of conservative physics and that of thermodynamics, was rather large.

It is the very basics of stress and deformation that are not in line with the systematics of physics. Hooke's law should have been a work term; the nature of stress is, before considering directional properties, first of all a change of state, the elastic potential; Euler's cut model is at variance with the definition of Newton's equilibrium conditions since the radius cannot be replaced by a surface orientation vector, and since a consideration of volume is missing altogether; Cauchy's continuity approach violates the existence theorem of the Gaussian potential theory. Nothing in continuum mechanics suggests the existence of strong internal bonds in solids against which work is to be done; but precisely this point is the explanation to the observation that a radius distance of a solid can act as a mechanical lever. In fact, a start of the theory with an equation of motion is an implicit statement that no bonds exist. The principles of the Euler-Cauchy theory apply to one single material only – infinitely slow, volume-neutral, equilibrium flow of a gas, such that no elastic potential builds up. The phenomenological approach to deformation is exemplified most evidently by the order in which the various subjects are tought to students: commonly textbooks start with an introduction to strain (the effect), continue with stress (the cause), and conclude with the material laws (the boundary conditions). In the rest of physics the development of thought is exactly in reverse order. It should be as in thermodynamics where first the equation of state is introduced, thus defining the material properties; the change of state is the cause for the effect, i.e. a change of volume as a function of the work done; the effect thus does not need a theory of its own, but it is the predicted result.

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